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# K-Theory Torsion

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**Abstract**

The Chern isomorphism determines the free part of the K-groups from ordinary cohomology. Thus to really understand the implications of K-theory for physics one must look at manifolds with K-torsion. Unfortunately there are not many explicit examples, and usually for very symmetric spaces. Cartesian products of $\mathbb{RP}^n$ are examples where the order of the torsion part differs between K-theory and ordinary cohomology. The dimension of corresponding branes is also discussed. An example for a Calabi-Yau manifold with K-torsion is given.

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# 1 Introduction

It has been shown [2] that K-theory classifies the topological charge of the D-brane gauge bundle (or the associated vector bundle). The crucial observation for this was that adding a brane-antibrane pair with the same gauge bundle does not change the total charge. Or in other words, you may add a D-p brane with the trivial bundle, then try to straighten out any "windings", in the bigger bundle, bring the bundle back in the form where the trivial bundle is one summand, and then remove the brane you added.

Mathematically, this is called stabilization, and the charge of the gauge bundle are stable isomorphism classes. It is not difficult to find examples of vector bundles that are stable isomorphic but not isomorphic, for example $TS^2$ and the trivial bundle $S^2 \times \mathbb{R}^2$ (as real vector bundles).

Another way to look at K-theory is that it is a generalized cohomology theory, that is it satisfies all the usual axioms except that higher cohomology groups of a point may not vanish. Since we can express usual field theory in terms of differential forms and de Rahm cohomology, it seems natural that a generalization of field theory leads to a generalized cohomology theory.

Now for all well-behaved spaces $X$ (such as topological manifolds or finite CW complexes), $K(X)$ is a finitely generated abelian group, i.e. of the form $\mathbb{Z}^n \oplus \text{Torsion}$. Interestingly, torsion charges can appear. In ordinary field theory you could also have torsion in integral cohomology $H^*(X, \mathbb{Z})$, but physical fields must be represented by differential forms, and this prohibits torsion. But on the K-theory side torsion charges are apparently physical charges. The purpose of this paper is to better understand the relation between integral cohomology and K-theory.

**For concreteness, I will restrict myself in the following to IIB string theory with spacetime manifold $X$, where the possible D-brane charges are $K(X)$, the Grothendiek group of complex vector bundles.**

Of course we want $X$ to be a 10–manifold (where Poincaré duality holds). In the following I will investigate compact topological manifolds of lower dimension which exhibit torsion in cohomology. The physical motivation for this is a spacetime of the form $M_d \times \mathbb{R}^{1,9-d}$, which on the K-theory side is just the $(10-d)$-th suspension of $M_d$. So the torsion of K-theory comes purely from the compact dimensions, and not from the Minkowski part of spacetime.

During the preparation of this paper another work appeared that also discusses the implications of the Atiyah–Hirzebruch spectral sequence [1].

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Here is a quick outline of the following sections:

1. This introduction

2. The Chern isomorphism and the Atiyah–Hirzebruch spectral sequence, which are the main tools used in this paper, are introduced. To demonstrate their utility I prove that the order of the torsion part is the same in integer cohomology and K-theory for odd $\mathbb{R}\mathrm{P}^n$. The sequence provides a necessary criterion for K-theory torsion.

3. Given some K-theory element, I determine the dimensionality of the corresponding D-brane (That is the minimum dimension needed to carry the charge). This can also be calculated from the Atiyah–Hirzebruch spectral sequence.

4. An example with different order of the torsion part in integer cohomology an K-theory is analyzed in detail.

5. By considering line bundles I find a sufficient criterion for K-theory torsion. This makes it possible to give an example for a Calabi-Yau manifold with torsion.

6. Conclusion

# 2 From $H^*(X, \mathbb{Z})$ to $K^*(X)$

The most important result is the Chern isomorphism:

$$\begin{aligned} K^0(X) \otimes_{\mathbb{Z}} \mathbb{R} &\simeq \bigoplus H^{2i}(X, \mathbb{R}) \stackrel{\text{def}}{=} H^{\text{ev}}(X, \mathbb{R}) \\ K^1(X) \otimes_{\mathbb{Z}} \mathbb{R} &\simeq \bigoplus H^{2i+1}(X, \mathbb{R}) \stackrel{\text{def}}{=} H^{\text{odd}}(X, \mathbb{R}) \end{aligned} \tag{1}$$

which is induced by the Chern character $\mathrm{ch} : K(X) \to H^{\text{ev}}(X, \mathbb{R})$.

This means that we can compute the free part of K-theory directly from ordinary de Rahm cohomology. Or in physical language, K-theory without torsion is just a reformulation of what one already knows from calculations on the level of differential forms. On the other hand side the torsion part of $H^{\text{ev}}(X, \mathbb{Z})$ and $K(X)$ do in general differ, for example[1]

$$\begin{aligned} K(\mathbb{R}\mathrm{P}^5) &= \mathbb{Z} \oplus \mathbb{Z}_4 \\ H^{\text{ev}}(\mathbb{R}\mathrm{P}^5, \mathbb{Z}) &= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{aligned} \tag{2}$$

[1] $\mathbb{R}\mathrm{P}^5$ is not spin, and therefore not a phenomenologically viable background spacetime. $\mathbb{R}\mathrm{P}^7$ would be a counterexample that is spin.

Remember the relevant Stiefel-Whitney classes $w_1(\mathbb{R}\mathrm{P}^n) = n + 1$, $w_2(\mathbb{R}\mathrm{P}^n) = \frac{n(n+1)}{2} \pmod 2$

It has been noted [5] that — although there is no surjective group homomorphism — the order of the torsion part is equal. Unfortunately, this is caused by peculiarities in the cohomology of real projective spaces and not a generic feature. A counterexample will be presented in section 4.

For now, lets use the Atiyah–Hirzebruch spectral sequence [8] to understand why the order is indeed equal for $\mathbb{RP}^n$, with $n$ odd so that the manifold is orientable. This spectral sequence stems from the filtration of the space by its CW-skeleton. It has the second term

$$E_2^{p,q} = \begin{cases} H^p(\mathbb{RP}^n, \mathbb{Z}) & q \text{ even} \\ 0 & q \text{ odd} \end{cases} \tag{3}$$

and converges towards the associated graded complex of $K(\mathbb{RP}^n)$.

For simplicity, take $n = 5$:

$$E_2^{p,q} = \begin{array}{r|cccccc} \uparrow & 0 & 0 & 0 & 0 & 0 & 0 \\ q & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ \hline & & & & & p \rightarrow & \end{array} \tag{4}$$

The differential $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ either has domain or range 0, thus

$$E_3^{p,q} = \ker d_2^{p,q} / \operatorname{img} d_2^{p-2,q+1} = E_2^{p,q} \tag{5}$$

In this sequence the $d_{\text{even}}$ are obviously irrelevant, and $E_{2k} = E_{2k+1}$.

The only[2] $d_3^{p,q} : E_3^{p,q} \to E_3^{p+3,q-2}$ with nonvanishing domain and range is[3] $d_3^{2,2} : \mathbb{Z}_2 \to \mathbb{Z}$. Since there is no nonzero group homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}$, $d_3 = 0$.

So far we found $E_5 = E_2$, and again there is only one $d_5$ with nonvanishing domain and range, $d_5^{0,4} : \mathbb{Z} \to \mathbb{Z}$. But the Chern isomorphism tells us that after tensoring everything with $\mathbb{Q}$ the spectral sequence already degenerates at level 2. Thus $d_5 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, and since domain is torsion free this implies $d_5 = 0$ (This also proves that torsion in cohomology is necessary for torsion in K-theory).

Thus $E_\infty = E_2$, but this is not enough to compute $K(\mathbb{RP}^5)$. All that it tells us is that there is a filtration

$$\begin{aligned} K^0(\mathbb{RP}^5) &= F_6^0 \supset F_5^0 \supset F_4^0 \supset F_3^0 \supset F_2^0 \supset F_1^0 \supset 0 \\ K^1(\mathbb{RP}^5) &= F_6^1 \supset F_5^1 \supset F_4^1 \supset F_3^1 \supset F_2^1 \supset F_1^1 \supset 0 \end{aligned} \tag{6}$$

[2] Of course the table is cyclic of order 2 in $q$

[3] In [8] it is noted without proof that $d_3 = Sq^3$, the third Steenrod Square

such that the successive cosets are the even respectively odd diagonals of $E_\infty$:

$$\begin{array}{cccccc} F_6^0/F_5^0=\mathbb{Z} & F_5^0/F_4^0=0 & F_4^0/F_3^0=\mathbb{Z}_2 & F_3^0/F_2^0=0 & F_2^0/F_1^0=\mathbb{Z}_2 & F_1^0/0=0 \\ F_6^1/F_5^1=0 & F_5^1/F_4^1=0 & F_4^1/F_3^1=0 & F_3^1/F_2^1=0 & F_2^1/F_1^1=0 & F_1^1/0=\mathbb{Z} \end{array} \tag{7}$$

Obviously $K^1(\mathbb{R}\mathrm{P}^5) = F_6^1 = F_5^1 = F_4^1 = F_3^1 = F_2^1 = F_1^1 = \mathbb{Z}$. But for $K^0(\mathbb{R}\mathrm{P}^5)$ we find $F_1^0 = 0$, $F_3^0 = F_2^0 = \mathbb{Z}_2$ and then hit the extension problem: either $\mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$ or $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)/\mathbb{Z}_2 = \mathbb{Z}_2$. So $F_5^0 = F_4^0 = \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $F_5^0$ is pure torsion each possibility determines a unique $K(\mathbb{R}\mathrm{P}^5) = F_6^0$, either $\mathbb{Z} \oplus \mathbb{Z}_4$ or $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

However, the ambiguity is between groups of equal order (since the ambiguous extension was between finite abelian groups), and moreover one of the possibilities was $H^{\mathrm{ev}}(\mathbb{R}\mathrm{P}^5)$, since the spectral sequence already degenerates at $E_2$.

The same argument can be used for all real projective spaces to prove that the order of the torsion part of cohomology and K-theory are equal, but as we have seen the proof depends on the special properties of $\mathbb{R}\mathrm{P}^n$.

# 3 Dimension of D-branes

## 3.1 Filtering $K(X)$

In flat space one can explicitly construct vector bundles that carry a nontrivial topological charge (using the Clifford algebra, see [2] and [4]). The bundle is trivial everywhere except on a hyperplane of even codimension, which is identified with the D-brane. One can extend this construction to general submanifolds with $\mathrm{Spin}_c$ normal bundle. This fits nicely to the fact that D-branes in IIB are even dimensional.

But to understand what the charges are one should rather understand which submanifold can carry a given K-theory element. The intuitive answer would be: An arbitrary submanifold $Y \subset X$ can carry the charge $x = [E] - [F] \in K(X)$ if there exists an isomorphism $E|_{X-Y} \simeq F|_{X-Y}$. Of course this is not well-defined, since the same K-theory element could be represented by different vector bundles $E'$, $F'$ that are stably isomorphic but not isomorphic. So we should really ask whether there exists an isomorphism $(E|_{X-Y} \oplus \mathbb{C}^k) \simeq (F|_{X-Y} \oplus \mathbb{C}^k)$ for some $k \in \mathbb{Z}$.

One would like to use the inclusion map $i : X - Y \hookrightarrow X$ to pull back $x$, and thus automatically include stabilization as an element of $K(X-Y)$, but unfortunately in general $i^*(E) - i^*(F) \notin K(X-Y)$ since the complement $X-Y$ is not compact (Remember that K-theory on noncompact spaces are differences of vector bundles that are isomorphic outside a compact subset).

So instead take compact submanifolds $j : Z \hookrightarrow X$ as probes: If their dimension is too low, they will generically miss the D-brane and the pullback $j^*(x) = 0 \in K(Z)$. Since $j^*$ depends only on the homotopy class of $j$, we do not have to worry about degenerate cases. If we cannot detect $x$ with submanifolds of a given dimension $p$ then we conclude that $x$ is carried by a D-brane of codimension greater than $p$.

But the total charge $0 \in K(X)$ could also be carried by a brane-antibrane pair that is separated in spacetime. Probing only in the neighborhood of one brane one would falsely find a charge. So our probe submanifold must somehow be big enough. Discussing this in terms of submanifolds is very cumbersome, so instead think of spacetime $X$ as a cell complex (simplicial complex or CW complex). Then take the $p$-skeleton $X^p$ as probe; it can easily be seen that this is independent of the chosen cell structure. Any cell complex embedded in $X$ is a subcomplex for some cell structure on $X$, in that sense $X^p$ probes the whole space.

Let $K_p(X)$ be the subgroup of $K(X)$ of charges that live on a brane[4] of codimension $p$ or higher, that is D-$(\dim(X) - p - 1)$-branes or lower. According to the previous arguments

$$K_p(X) = \ker\Big(K(X) \to K(X^{p-1})\Big) \tag{8}$$

where the map is the one induced by the inclusion $X^{p-1} \hookrightarrow X$.

This yields a filtration

$$K(X) = K_0(X) \supset \tilde{K}(X) = K_1(X) \supset K_2(X) \supset \cdots \supset K_{\dim X+1}(X) = 0 \tag{9}$$

where the successive quotients $K_p(X)/K_{p+1}(X)$ are the D-$(\dim(X) - p - 1)$-brane charges.

## 3.2 Remarks

Lets try to understand eq. (8) better. $K_{\dim X+1}(X) = 0$ means that there are no D-$(-2)$-branes or less, which is correct.

The 9-brane charges ($p = 0$) are $K(X)/\tilde{K}(X) = \mathbb{Z}$, which is the virtual rank of the bundle pair. This we can also understand: If we do not start with the same number of 9 and $\bar{9}$-branes, then there will always be a 10-dimensional brane left. On the other hand side if the virtual rank is 0 (as required by tadpole cancellation), then the vector bundles are isomorphic over sufficiently small open sets (since they are locally trivial), which one could use to localize the nontrivial windings at a subspace of codimension 1.

Fortunately there is a way to calculate the quotients $K_p(X)/K_{p+1}(X)$. First note that one can extend eq. (8) to the higher $K$-groups straightforwardly:

$$K_p^n(X) = \ker\Big(K^n(X) \to K^n(X^{p-1})\Big) \tag{10}$$

[4]More precisely a stack of coincident branes, although I will not make that distinction in the following

And the associated graded complex to this filtration is precisely the limit of the Atiyah–Hirzebruch spectral sequence

$$E_\infty^{p,q} = K_p^{p+q}(X)/K_{p+1}^{p+q}(X) \tag{11}$$

If there is no torsion in integer cohomology then the spectral sequence degenerates at level 2, and (compare eq. (3))

$$K_p^*(X)/K_{p+1}^*(X) \simeq H^p(X,\mathbb{Z}) \tag{12}$$

where the isomorphism is just the Chern character. This confirms the interpretation of the dimensionality of the K-theory elements.

The odd rows in $E_\infty^{p,q}$ all vanish, so for odd $p$ and odd $q$

$$K_p^{p+q}(X)/K_{p+1}^{p+q}(X) = K_p(X)/K_{p+1}(X) = 0 \tag{13}$$

which just means that there are no topological charges for odd dimensional D-branes.

A word of caution: even if $K(X)$ is torsion-free, one of the successive quotients can be torsion, as in the example $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Physically, this means that there can be an apparent torsion charge on a D-brane in the sense that multiple copies of that brane can decay to something lower-dimensional, which a single brane cannot. But the lower-dimensional remnant then carries an ordinary (non-torsion) charge that keeps track of the number of branes we started with.

# 4 Examples

## 4.1 $K(\mathbb{R}\mathrm{P}^n)$

The best way to construct manifolds with K-torsion is to use quotients of well-understood manifolds (like the sphere) by free group actions. At the example $\mathbb{R}\mathrm{P}^n$ I will review the necessary tools (See e.g. [7]).

Let $\mathbb{Z}_2$ act on $S^n$ ($n$ odd) via the antipodal map, a free group action. In general (for free group actions) K-theory on the quotient is equal to equivariant K-theory on the covering space $K^*(S^n/\mathbb{Z}_2) = K^*_{\mathbb{Z}_2}(S^n)$. Writing down the (cyclic) long exact sequence associated to the inclusion $S^n \hookrightarrow D^{n+1}$, we find:

$$\begin{array}{ccccc} K^1_{\mathbb{Z}_2}(S^n) & \longleftarrow & K^1_{\mathbb{Z}_2}(D^{n+1}) & \longleftarrow & K^1_{\mathbb{Z}_2}(D^{n+1},S^n) \\ \downarrow & & & & \uparrow \\ K^0_{\mathbb{Z}_2}(D^{n+1},S^n) & \longrightarrow & K^0_{\mathbb{Z}_2}(D^{n+1}) & \longrightarrow & K^0_{\mathbb{Z}_2}(S^n) \end{array} \tag{14}$$

where the $\mathbb{Z}_2$ action on the disk $D^{n+1}$ is the obvious extension of the $\mathbb{Z}_2$–action on $S^n$.

Now identify $K^0_{\mathbb{Z}_2}(D^{n+1}, S^n)$, virtual differences of vector bundles on $D^{n+1}$ that are isomorphic over the boundary, with $K^0_{\mathbb{Z}_2}(\mathbb{R}^{n+1})$, virtual differences on $\mathbb{R}^{n+1}$ with isomorphism outside a compact subset. The associated $\mathbb{Z}_2$–action on $\mathbb{R}^{n+1}$ is again $x \mapsto -x$. Since $n+1$ is even, we can interpret $\mathbb{R}^{n+1} = \mathbb{C}^{(n+1)/2} \stackrel{\text{def}}{=} \mathbb{C}^m$ with a linear $\mathbb{Z}_2$–action on $\mathbb{C}^m$. And this is a $\mathbb{Z}_2$–equivariant vector bundle over a point.

Then use the Thom isomorphism, that is $K_G(E) = K_G(X)$ for any $G$–vector bundle $E$ over $X$ (as abelian groups, the multiplication law is different):

$$K^0_{\mathbb{Z}_2}(D^{n+1}, S^n) = K^0_{\mathbb{Z}_2}(\mathbb{C}^m) = K^0_{\mathbb{Z}_2}(\{\text{pt}\}) = R(\mathbb{Z}_2) = \mathbb{Z}[x]/x^2 - 1 \tag{15}$$

$R(\mathbb{Z}_2)$ are the formal differences of representations of $\mathbb{Z}_2$ (with the obvious ring structure induced by the tensor product of representations), and $x$ denotes the unique nontrivial irreducible representation of $\mathbb{Z}_2$. If one is only interested in the underlying abelian group, this is of course $\mathbb{Z} \oplus \mathbb{Z}$.

Doing the same for $K^1$ and using the homotopy $D^{n+1} \sim \{pt\}$, we evaluate eq. (14):

$$\begin{array}{ccccc} K^1_{\mathbb{Z}_2}(S^n) & \longleftarrow & 0 & \longleftarrow & 0 \\ \downarrow & & & & \uparrow \\ \mathbb{Z}[x]/x^2 - 1 & \xrightarrow{f} & \mathbb{Z}[x]/x^2 - 1 & \xrightarrow{g} & K^0_{\mathbb{Z}_2}(S^n) \end{array} \tag{16}$$

Since $\mathbb{Z}[x]/x^2 - 1$ is torsion free as abelian group, so must be $K^1_{\mathbb{Z}_2}(S^n) = K^1(\mathbb{R}\mathrm{P}^n)$. From the Chern isomorphism then follows that $K^1(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}$. But to determine the torsion part of $K^0(\mathbb{R}\mathrm{P}^n)$, we need to identify the map $f$. Tracing everything back to the Thom isomorphism, one can show that $f$ is multiplication with $(x-1)^m$. Using exactness ($\operatorname{img} f = \ker g$) we find

$$\begin{aligned} K^0(\mathbb{R}\mathrm{P}^n) &= K^0_{\mathbb{Z}_2}(S^n) = \mathbb{Z}[x]/\left\langle x^2 - 1, (x-1)^m \right\rangle = \\ &= \mathbb{Z}[z]/\left\langle (z+1)^2 - 1, z^m \right\rangle = \mathbb{Z}[z]/\left\langle z^2 + 2z, z^m \right\rangle \end{aligned} \tag{17}$$

Up to the given relations, each ring element can be represented as $az + b$, $a, b \in \mathbb{Z}$. While $b$ is not subject to any relation, we can use $z^2 + 2z = 0$ and $z^m = 0$ to show $2^{m-1}z = 0$. Therefore (ignoring the ring structure):

$$\begin{aligned} K^1(\mathbb{R}\mathrm{P}^n) &= \mathbb{Z} \\ K^0(\mathbb{R}\mathrm{P}^n) &= \mathbb{Z} \oplus \mathbb{Z}_{2^{m-1}} \end{aligned} \tag{18}$$

## 4.2 $K(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5)$

Here is the promised example of a space where the order of the torsion subgroup in K-theory and ordinary cohomology differs.

Of course we use the Künneth formula to calculate the cohomology[5] of a Cartesian product:

$$0 \longrightarrow \bigoplus_{i+j=m} H^i(X) \otimes H^j(Y) \longrightarrow H^m(X \times Y) \longrightarrow \bigoplus_{i+j=m+1} \operatorname{Tor}\big(H^i(X), H^j(Y)\big) \longrightarrow 0 \quad (19)$$

The cohomology of real projective space is

$$H^i(\mathbb{R}\mathrm{P}^3) = \begin{cases} \mathbb{Z} & i=3 \\ \mathbb{Z}_2 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0 \end{cases} \qquad H^i(\mathbb{R}\mathrm{P}^5) = \begin{cases} \mathbb{Z} & i=5 \\ \mathbb{Z}_2 & i=4 \\ 0 & i=3 \\ \mathbb{Z}_2 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0 \end{cases} \quad (20)$$

Thus eq. (19) contains the exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H^8(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow & H^7(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & H^6(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}_2 & \longrightarrow & H^5(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow & H^4(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H^3(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow & H^2(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & H^1(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H^0(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \quad (21)$$

Using Poincaré duality ($\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5$ is an orientable 8-manifold since each factor is), $H^5_{\text{tors}} \simeq H^4_{\text{tors}}$ and $H^3_{\text{tors}} \simeq H^6_{\text{tors}}$. This fixes the extension ambiguities, and we find

$$H^i(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) = \begin{cases} \mathbb{Z} & i=8 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i=7 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & i=6 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i=5 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i=4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & i=3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0 \end{cases} \quad \Rightarrow \quad \begin{cases} H^{\text{ev}}(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) = \mathbb{Z}^2 \oplus \mathbb{Z}_2^5 \\ H^{\text{odd}}(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) = \mathbb{Z}^2 \oplus \mathbb{Z}_2^5 \end{cases} \quad (22)$$

[5] In this section, $H^*(X)$ is always cohomology with integer coefficients

For K-theory there is the following [9] analog to the ordinary Künneth formula:

$$0 \longrightarrow \bigoplus_{i+j=m} K^i(X) \otimes K^j(Y) \longrightarrow K^m(X \times Y) \longrightarrow \bigoplus_{i+j=m+1} \mathrm{Tor}\left(K^i(X), K^j(Y)\right) \longrightarrow 0 \quad (23)$$

where all indices are modulo 2. Thus

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Big[\mathbb{Z} \otimes (\mathbb{Z} \oplus \mathbb{Z}_4)\Big] \oplus \Big[(\mathbb{Z} \oplus \mathbb{Z}_2) \otimes \mathbb{Z}\Big] & \longrightarrow & K^1(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \\
 & & \shortparallel & & & & & & \\
 & & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 & & & & & & \\
0 & \longrightarrow & \Big[\mathbb{Z} \otimes \mathbb{Z}\Big] \oplus \Big[(\mathbb{Z} \oplus \mathbb{Z}_2) \otimes (\mathbb{Z} \oplus \mathbb{Z}_4)\Big] & \longrightarrow & K^0(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \shortparallel & & & & & & \\
 & & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & & & & &
\end{array} \quad (24)$$

Using the duality [6][6] between the torsion part of $K^0$ and $K^1$ for an even-dimensional orientable manifold, we arrive at the following result:

$$\begin{aligned}
K^1(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) &= \mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2 \\
K^0(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5) &= \mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2
\end{aligned} \quad (25)$$

The order of the torsion subgroups of $K^0$ and $H^{\mathrm{ev}}$ does not match. Tracing it back through our calculation, we see that this stems from the well-known fact that the order of the torsion of a tensor product is not determined by the orders of the torsion subgroups of the factors. To be precise $\mathbb{Z}_2 \otimes \mathbb{Z}_4 = \mathbb{Z}_2$, while $\mathbb{Z}_2 \otimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

### 4.3 Complete Intersections

For physical reasons it would be nice if the underlying space is Calabi-Yau. Unfortunately hypersurfaces in toric varieties have torsion-free K-theory:

The smooth toric variety (of complex dimension $m$) does not have torsion in integer homology. The Lefschetz hyperplane theorem yields torsion free homology of the hypersurface in (real) dimensions 0 to $m-1$. But Poincaré duality then fixes the torsion part of the whole homology, since $H_i(X, \mathbb{Z})_{\mathrm{tors}} \simeq H_{\dim X - 1 - i}(X, \mathbb{Z})_{\mathrm{tors}}$. Duality with integer cohomology then gives rise to torsion free cohomology. But torsion in integer cohomology is necessary for K-theory torsion.

## 5 Multiply Connected Spaces

We have seen that integer cohomology provides a necessary although not sufficient tool to determine whether a given manifold has torsion in K-theory. The purpose of this section is

[6] I am grateful to Ulrike Tillmann for sketching to me how one could give a rigorous proof

to give an easy sufficient criterion. The idea is that line bundles are stably isomorphic if and only if they are isomorphic, so stability is not a relevant concept for one-dimensional vector bundles. Then we just have to construct line bundles where a certain finite sum is (stable) trivial. This happens if the first Chern class $c_1 \in H^2(X, \mathbb{Z})$ is torsion.

## 5.1 Line Bundles

Let us have a closer look to the aforementioned properties of line bundles. A $n$-dimensional vector bundle is in general defined via its transition functions on some open cover $X = \cup_{i \in I} U_i$:

$$g_{ij} : U_i \cap U_j \to U(n, \mathbb{C}) \tag{26}$$

For a line bundle, this means

$$g_{ij} : U_i \cap U_j \to U(1) \tag{27}$$

Now two line bundles $L_1, L_2$ (with transition functions $g^{(1)}, g^{(2)}$) are stably isomorphic if there exists an $n \in \mathbb{N}$:

$$L_1 \oplus \mathbb{C}^n \simeq L_2 \oplus \mathbb{C}^n \tag{28}$$

But the determinant bundle of a line bundle plus a trivial bundle is again the line bundle. Remember that the transition functions $\tilde{g}_{ij}^{(k)}$ of the determinant bundle $\wedge^{n+1}(L_k \oplus \mathbb{C}^n)$ are the determinants of the transition function matrices of $L_k \oplus \mathbb{C}^n$:

$$\tilde{g}_{ij}^{(k)} = \det \begin{pmatrix} g_{ij}^{(k)} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = g_{ij}^{(k)} \qquad k = 1, 2 \tag{29}$$

Therefore

$$L_1 \oplus \mathbb{C}^n \simeq L_2 \oplus \mathbb{C}^n \quad \Rightarrow \quad \wedge^{n+1}(L_1 \oplus \mathbb{C}^n) \simeq \wedge^{n+1}(L_2 \oplus \mathbb{C}^n) \quad \Rightarrow \quad L_1 \simeq L_2 \tag{30}$$

Of course the "$\Leftarrow$" is trivial.

By a standard argument we identify then the isomorphism classes of transition functions of line bundles with the Cech cohomology group $H^1\left(X, C^0(U(1))\right)$, where $C^0(U(1))$ is the sheaf of $U(1)$-valued continuous functions. The long exact sequence associated to the exponential short (sheaf) exact sequence is then

$$\cdots \to H^1(X, C^0(\mathbb{R})) \to H^1\Big(X, C^0(U(1))\Big) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, C^0(\mathbb{R})) \to \cdots \tag{31}$$

which yields the desired isomorphism since $C^0(\mathbb{R})$ is a fine sheaf, $H^i(X, C^0(\mathbb{R})) = 0 \ \forall i \geq 1$.

## 5.2 Adding line bundles

Now assume $E$ is a line bundle on $X$ with $0 \neq c_1(E) \in H^2(X,\mathbb{Z})$ pure torsion (according to the previous section then $[E]-[1] \neq 0 \in K(X)$). But observe that the group law in $K(X)$ is based on the Whitney sum $E \oplus E$, while the group law in $H^2(X,\mathbb{Z})$ corresponds to the tensor product[7] $E \otimes E$. And of course $[E] \in K(X)$ does not generate a torsion subgroup since

$$\dim(n[E]) = \dim\Big(\underbrace{[E]+\cdots+[E]}_{n\ \text{times}}\Big) = n \neq 0 \qquad \forall n \in \mathbb{Z}-\{0\} \tag{32}$$

However $[E]-[1] \in K(X)$ is a torsion element (1 denotes the trivial line bundle). This follows from the Chern isomorphism:

**Corollary 1** Let $0 \neq x \in K(X)$. Then $x$ is a torsion element if and only if $\mathrm{ch}(x) = 0$.

**Proof 1**

- "$\Rightarrow$": Since $\mathrm{ch} : K(X) \to H^{\mathrm{ev}}(X,\mathbb{R})$ is a group homomorphism this is trivial.
- "$\Leftarrow$": Assume that $x \in K(X)$ is free but $\mathrm{ch}(x) = 0$. Thus $\dim(\mathrm{img}(\mathrm{ch})) < \mathrm{rk}(K(X))$, in contradiction to the Chern isomorphism (eq. (1)).

In our case the Chern character $\mathrm{ch}(E) = e^{c_1(E)} = 1 + c_1(E) + \cdots = 1 \in H^{\mathrm{ev}}(X,\mathbb{R})$ since $c_1(E)$ was assumed to be a torsion element in $H^2(X,\mathbb{Z})$ (so its image in $H^2(X,\mathbb{R})$ vanishes). Therefore $\mathrm{ch}([E]-[1]) = \mathrm{ch}(E) - \mathrm{ch}(1) = 0$ and $[E]-[1]$ generates a nontrivial torsion subgroup.

## 5.3 Multiply connected Calabi-Yau manifolds: Quintics

Consider the Fermat quintic $Y \subset \mathbb{C}\mathrm{P}^4$:

$$\sum_{i=1}^{5} z_i^5 = 0 \tag{33}$$

with the $\mathbb{Z}_5 = G = \{1, g, g^2, g^3, g^4\}$ symmetry generated by

$$g : \ \ [z_1 : z_2 : z_3 : z_4 : z_5] \mapsto \left[z_1 : \alpha z_2 : \alpha^2 z_3 : \alpha^3 z_4 : \alpha^4 z_5\right] \qquad \alpha = e^{\frac{2\pi i}{5}} \tag{34}$$

The group $G$ acts freely on $Y$: The only fixed point $[1:0:0:0:0] \in \mathbb{C}\mathrm{P}^4$ of the ambient space is missed by the hypersurface eq. (33).

This means that the quotient $X = Y/G$ is a (nonsingular) Calabi-Yau manifold. The quotient is still projective algebraic, but of course not a complete intersection since this

[7] By the isomorphism in eq. (31) this is the group law in $H^1(X, C^0(U(1)))$, which corresponds to multiplying the $U(1)$ transition functions

would contradict section 4.3; this simply means that it is a hypersurface in some projective space where one cannot eliminate all equations.

Since the quintic $Y$ was simply connected (as every complete intersection), we can determine the quotient's fundamental group from the long exact homotopy sequence (for $Y$ as a bundle over $X$ with fiber $G$):

$$\cdots \;\to\; \underbrace{\pi_1(G)}_{=0} \;\to\; \underbrace{\pi_1(Y)}_{=0} \;\to\; \pi_1(X) \;\to\; \underbrace{\pi_0(G)}_{=G} \;\to\; \underbrace{\pi_0(Y)}_{=0} \;\to\; \underbrace{\pi_0(X)}_{=0} \tag{35}$$

Since $Y$ is a complete intersection, $h^{1,1}(Y) = h^{1,1}(\mathbb{C}\mathrm{P}^4) = 1$. The quotient $X$ is still Kähler (the Kähler class $\omega = \partial\bar{\partial}\log||Z||^2$ is $G$-invariant), so that $h^{1,1}(X) = h^{1,1}(Y) = 1$.

The complex structure deformations $h^{2,1}(Y)$ correspond to the monomials modulo $\mathrm{PGL}(4)$ (the automorphisms of the ambient space) and rescaling of the equation. Here there are $\binom{5+5-1}{5} = 126$ monomials, and $|\,\mathrm{PGL}(4)| = 24$. Therefore $h^{2,1}(Y) = 126-24-1 = 101$. The complex structure deformations of the quotient are the $G$-invariant monomials, straightforward counting gives 26. But now by treating every coordinate separately in the $G$-action the full $\mathrm{PGL}(4)$ is broken to the diagonal subgroup (4 parameters). Therefore $h^{2,1}(X) = 26-4-1 = 21$. This is confirmed by the Euler number

$$\chi(Y) = 2\Big(h^{2,1}(Y) - h^{1,1}(Y)\Big) = 200 \qquad \chi(X) = 2\Big(h^{2,1}(X) - h^{1,1}(X)\Big) = 40 \tag{36}$$

As we expect for a free $\mathbb{Z}_5$ group action, $\chi(Y) = 5\chi(X)$. The Hodge diamond

$$h^{p,q}(X) = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 21 & & 21 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{37}$$

determines the free part of integer cohomology, now we have to find the torsion part. For every manifold $H^1(X,\mathbb{Z})$ is torsion free, since the torsion part is dual to the torsion part in $H_0(X,\mathbb{Z}) = \mathbb{Z}$. Furthermore $H_1(X,\mathbb{Z})$ is the abelianization of $\pi_1(X) = \mathbb{Z}_5$, which was already abelian. Therefore $H_1(X,\mathbb{Z}) = \mathbb{Z}_5$. By the universal coefficient theorem $H^2(X,\mathbb{Z})_{\text{tors}} \simeq H_1(X,\mathbb{Z})_{\text{tors}} = \mathbb{Z}_5$.

The hard part is the torsion in $H^3$ (Poincaré duality then determines the rest). We are going to use the following sequence [12]:

$$0 \;\to\; \Sigma_2 \;\to\; H_2(X,\mathbb{Z}) \;\to\; H_2(\mathbb{Z}_5) \;\to\; 0 \tag{38}$$

where[8] $\Sigma_2$ is the image of $\pi_2(X)$ in $H_2(X,\mathbb{Z})$. With other words $\Sigma_2$ are the homology classes that can be represented by 2–spheres.

So we need to determine $\pi_2(X)$ first. We know that on the covering space $\pi_2(Y) = H_2(Y) = \mathbb{Z}$ (The Hurewicz isomorphism theorem) since $Y$ is simply connected. But every map $f : S^2 \to X$ can be lifted to $\tilde{f} : S^2 \to Y$ since the $S^2$ is simply connected. That is the $S^2$ cannot wrap the nontrivial $S^1 \subset X$. More formally we can use the homotopy long exact sequence:

$$\cdots \;\to\; \underbrace{\pi_2(G)}_{=0} \;\to\; \pi_2(Y) \;\to\; \pi_2(X) \;\to\; \underbrace{\pi_1(G)}_{=0} \;\to\; \cdots \tag{39}$$

to show that $\pi_2(X) = \pi_2(Y) = \mathbb{Z}$.

The group homology $H_2(\mathbb{Z}_5) = 0$, therefore eq. (38) determines an isomorphism $\Sigma_2 \simeq H_2(X,\mathbb{Z})$. We know already that the free part $H_2(X,\mathbb{Z})_{\text{free}} = \mathbb{Z}$ from the Hodge diamond. But then the map $\pi_2(X) \to \Sigma_2$ must have been injective since the domain is $\mathbb{Z}$ and the image at least $\mathbb{Z}$. Therefore $\Sigma_2 = \mathbb{Z}$ and the torsion part $H^3(X,\mathbb{Z})_{\text{tors}} \simeq H_2(X,\mathbb{Z})_{\text{tors}} = 0$.

We have seen that

$$H^i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=6 \\ \mathbb{Z}_5 & i=5 \\ \mathbb{Z} & i=4 \\ \mathbb{Z}^{44} & i=3 \\ \mathbb{Z}\oplus\mathbb{Z}_5 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0 \end{cases} \tag{40}$$

From the Atiyah–Hirzebruch spectral sequence it is obvious that either the $\mathbb{Z}_5$ torsion part survives to K-theory or vanishes (there is no subgroup except the trivial group). But according to the previous section there exists a torsion subgroup. Therefore $K(X)_{\text{tors}} = \mathbb{Z}_5$. Using Chern isomorphism and duality, this determines K-theory completely:

$$K^i(X) = \begin{cases} \mathbb{Z}^{44}\oplus\mathbb{Z}_5 & i=1 \\ \mathbb{Z}^4\oplus\mathbb{Z}_5 & i=0 \end{cases} \tag{41}$$

## 5.4 The Tian–Yau manifold

The first known example was the the Tian–Yau threefold $X = Y/\mathbb{Z}_3$, a complete intersection $Y$ in $\mathbb{CP}^3 \times \mathbb{CP}^3$ with a free $\mathbb{Z}_3$ group action.

[8] This is corrected version of the sequence in [11]

The Hodge diamond can be found via counting monomials (see [10] for details):

$$h^{p,q}(Y) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 14 & & 0 & \\ 1 & & 23 & & 23 & & 1 \\ & 0 & & 14 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad h^{p,q}(X) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 6 & & 0 & \\ 1 & & 9 & & 9 & & 1 \\ & 0 & & 6 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{42}$$

Since the fundamental group of the quotient $\pi_1(X) = \mathbb{Z}_3$ we also know the torsion part $H_1(X, \mathbb{Z}) = \mathbb{Z}_3$. It remains to determine the torsion part of $H^3(X, \mathbb{Z})_{\text{tors}} \simeq H^4(X, \mathbb{Z})_{\text{tors}} \simeq H_2(X, \mathbb{Z})_{\text{tors}}$.

But in this case the sequence eq. (38) does not suffice. Again the group homology $H_2(\mathbb{Z}_3) = 0$, but now it is unclear what $\Sigma_2$ is. All we know is $\pi_2(X) = \pi_2(Y) = \mathbb{Z}^{14}$, and the free part $H_2(X, \mathbb{Z})_{\text{free}} = \mathbb{Z}^6$. There is no reason for the image $\pi_2(X) \to \Sigma_2$ not to contain a torsion part.

Therefore

$$H^i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 6 \\ \mathbb{Z}_3 & i = 5 \\ \mathbb{Z}^6 \oplus T & i = 4 \\ \mathbb{Z}^{20} \oplus T & i = 3 \\ \mathbb{Z}^6 \oplus \mathbb{Z}_3 & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0 \end{cases} \tag{43}$$

where $T$ is the unknown torsion part. The Atiyah–Hirzebruch spectral sequence not only kills torsion subgroups in cohomology, it also puts them together differently via extensions. So we know that K-theory has torsion, but cannot determine the groups.

# 6 Conclusion

As we have explicitly seen the order of K-theory-torsion and cohomology torsion is in general different. Thus substituting integer cohomology for K-theory not only leads to the wrong charge addition rules, it also does not yield the correct number of charges. Although not being totally independent, one must consider the whole spectral sequence connecting them.

This implies that discrete torsion on the field theory level must be different from the K-theory interpretation of D-brane charges. The most promising idea for a complete treatment

is trying to find a pairing (preferably a perfect pairing) between K-theory and something else (maybe again K-theory) to $U(1)$ and use this to construct a suitable partition function, as in [3][6].

The whole discussion might be even relevant to the real world, since phenomenologically interesting string compactifications need finite non-zero $H_1(X,\mathbb{Z})$ in order to further break the gauge group via Wilson lines. But by the universal coefficient theorem, $H^2(X,\mathbb{Z})_{\text{tors}} \simeq H_1(X,\mathbb{Z})_{\text{tors}}$, so torsion charges appear in all realistic compactifications.

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